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# Staggered grid implementation of 1D Boussinesq model for simulating dispersive wave

**D Adytia[1, a)] , D Tarwidi[1], S A Kifli[1] and S R Pudjaprasetya[2],**

[1]School of Computing, Telkom University, Jl. Telekomunikasi 1 Bandung 40257, Indonesia
[2]Industrial & Financial Mathematics Research Group, Faculty of Mathematics & Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha 10, Bandung 40132, Indonesia

adytia@telkomuniversity.ac.id

**Abstract**. In this paper, a numerical implementation of 1D Variational Boussinesq (VB) wave model in a staggered grid scheme is discussed. The staggered grid scheme that is used is based on the idea proposed by Stelling & Duinmeijer (2003) who implemented the scheme in a non-dispersive Shallow Water Equations in a conservative form. Here, we extend the idea of the staggered scheme to be applied for VB wave model. To test the accuracy of the implementation, we test the numerical implementation of VB wave model for simulating propagation of solitary wave against analytical solution. Moreover, to test dispersiveness of the model, we simulate a standing wave against analytical solution. Results of simulations show a good agreement with analytical solutions.

## 1. Introduction

Water wave modelling has been a favorite topic among coastal engineers since the last 3 decades; especially when it is used as a tool to understand wave propagation and behavior for design engineering of an offshore as well as coastal structure. Over several physical aspects of water wave, the most import aspects are dispersion property and nonlinearity. Dispersion in water wave means that longer wave (or low frequency waves) travels faster than shorter waves (high frequency waves). Boussinesq-type models (BTMs) are the most favorite wave model for researchers to simulate wave propagation, for long wave such as tsunami, as well as short wave such as wind wave. The original Boussinesq model was introduced by J.V. Boussinesq in 1872 [7] is valid only for long waves above flat bottom. The original model is then extended by Peregrine in 1967 [14] to be able to simulate wave above an uneven bottom. Two most popular BTMs are the Boussinesq of Madsen & Sorensen in 1992 [12] and the Boussinesq of Nwogu in 1993 [13]. Since then, these BTMs are extended for simulating highly dispersive and highly nonlinear wave propagation; see [8] for review of the development of BTMs.

Numerical implementation to be chosen for the BTMs is also important to solve the model numerically. Most popular numerical implementations among these BTMs are Finite Difference Method (FDM), Finite Volume Method (FVM) and Finite Element Method (FEM). These numerical implementations are mostly using collocated grid, i.e. all variables are defined in the same grid points. For continuous problems the collocated grid gives good performance, while in boundary condition

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such wet-dry condition, the collocated grid may lead to a stability problem where an artificial damping is required (see [11]). A different approach is a staggered grid approach, where grids of surface elevation and velocity are in different locations. Stelling & Duinmeijer in 2003 [15] propose a staggered grid scheme for solving Shallow Water Equations (SWE) that performs relative good for rapidly varied shallow water flows. The staggered scheme is then extended for simulating dispersive Non-Hydrostatic wave model in [16].

In this paper, we use a Boussinesq type model that is derived by using a variational approach, i.e. that is so-called the Variational Boussinesq (VB) model. The model was implemented by using spectral method in [9] and by using Finite Element Method (FEM), see [1, 2, 4, 5, 6] and later is extended to a fully nonlinear model by also using FEM [3]. Both implementations are using collocated grid, which give challenges when dealing with wet-dry boundary conditions. Here, we implemented numerically the VB model in a staggered grid scheme as proposed by Stelling & Duinmeijer in [15]. Accuracy of the implementation is tested for simulating two cases, i.e. a dispersive standing wave in a close basin and a propagation of a solitary wave.

The structure of this paper is as follows. In the next section, we describe the VB model as it is introduced in [1]. This section is followed by a staggered grid implementation of the VB model using the basic idea in [15]. In Section 4, we test the performance of the implementation for simulation two test cases against analytical solution. Finally, some conclusions and discussions are described in the last section.

## 2. Variational Boussinesq Model

In this paper, we use the 1D Variational Boussinesq (VB) model that is introduced by Klopman ea. in 2010 [9], for simulating dispersive wave. The VB model is derived based on variational formulation proposed by Luke in 1967 [10]. By assuming the water as an ideal fluid, i.e. inviscid, incompressible and the flow is assumed to be irrotational, the dynamic of wave can be exactly discribed by a Hamiltonian system, see [17]. The Hamiltonian is the total energy, i.e. sum of the potential and the kinetic energy. As derived in [1], the 1D VB model is described by two dynamic equations as follows

$$\partial_t\eta = -\partial_x\big((h+\eta)\partial_x\phi\big) - \partial_x(\beta\partial_x\psi) \qquad (2.1)$$

$$\partial_t\phi = -g\eta - (\partial_x\phi)^2/2 \qquad (2.2)$$

and an additional elliptic equation

$$-\partial_x(\alpha\partial_x\psi) + \gamma\psi = \partial_x(\beta\partial_x\phi) \qquad (2.3)$$

where $\eta(x,t)$ and $\varphi(x,t)$ are the canonical variables of the Hamiltonian system, i.e. surface elevation and surface potential, respectively. Here, *h(x)* is the water depth and $g = 9.81 m/s^2$ is the gravitational acceleration. A spatially dependent variable $\psi(x)$ is an auxiliary variable that is used in the approximation of the VB model, see [1, 5], which has to calculated in every time step by solving the elliptic equation in (2.3). The coefficients $\alpha, \beta$ & $\gamma$ are the VB coefficients. For parabolic vertical profile the coefficients are given by

$$\alpha = \frac{2}{15}H^3; \quad \beta = -\frac{H^2}{3}; \quad \gamma = \frac{H}{3} \qquad (2.4)$$

where $H = h + \eta$ is the total depth.

Note that the VB model in (2.1)-(2.3) are written in canonical variables $\eta(x,t)$ and $\varphi(x,t)$. Most Boussinesq type of model as well as non-hydrostatic model are written in $\eta(x,t)$ and $u(x,t)$, where *u(x,t)* is horizontal velocity. As shown in [9], the VB model can be written in surface horizontal velocity *u(x,t)* by using the fact that $u$ can be written as $u = \partial_x\phi$. Therefore, the VB model can be rewritten as

$$\partial_t\eta = -\partial_x(Hu) - \partial_x(\beta\partial_x\psi) \qquad (2.5)$$

$$\partial_t u = -g\partial_x\eta - u\partial_x u \qquad (2.6)$$

$$-\partial_x(\alpha\partial_x\psi) + \gamma\psi = \partial_x(\beta u) \qquad (2.7)$$

Notice that equation (2.5) and (2.7) are actually the Shallow Water Equations (SWE) with an additional term in the continuity equation (2.5), i.e. $-\partial_x(\beta\partial_x\psi)$, and the equation (2.6) is an additional equation for searching $\psi$ in every time step. As described in [3] the VB model (2.4)-(2.6) can be extended for simulating highly dispersive and strongly nonlinear waves. In the next section we describe the staggered grid scheme that is proposed by [15] for implementing numerically the equation (2.4)-(2.6).

## 3. Staggered Grid Scheme

To design an efficient and accurate wave code, not only an accurate wave model is needed but it is also important to choose a stable, simple, and efficient numerical implementation. In this paper we implemented the VB model in (2.4)-(2.6) numerically in a staggered grid scheme proposed in [15]. Staggered grid means that eq. (2.4) and (2.5) are approximated on different cells. Here, $\eta$ and $\psi$ are calculated at full grid points notated by $x_i,\ i = 1, 2, \cdots, N$ and $u$ is calculated at half grid points $x_{i+1/2}$, $i = 1, 2, \cdots, N + 1$ as illustrated in Figure 1. Just as $\eta$, the depth $h$ and the total depth $H$ is also in the full grid. As consequences, the coefficient $\alpha, \beta$ and $\gamma$ are also in the full grid.

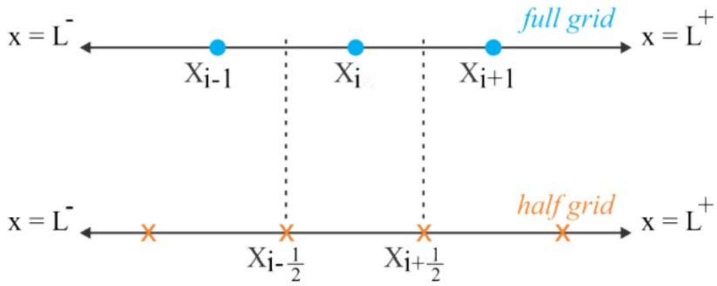


**Figure. 1.** Illustration of staggered grid scheme.

In the staggered scheme discretization proposed by Stelling & Duinmeijer (2003) [15], the spatial derivatives are approximated by second order center difference and first order upwind scheme. The approximation of the continuity equation of VB model (2.5) is given by the following relation

$$\frac{\eta_i^{n+1}-\eta_i^n}{\Delta t} = -\left(\frac{{}^*H^n_{i+\frac{1}{2}}u^n_{i+\frac{1}{2}} - {}^*H^n_{i-\frac{1}{2}}u^n_{i-\frac{1}{2}}}{\Delta x}\right) - \left(\frac{{}^*\beta^n_{i+\frac{1}{2}}(\psi^n_{i+1}-\psi^n_i) - {}^*\beta^n_{i-\frac{1}{2}}(\psi^n_i-\psi^n_{i-1})}{(\Delta x)^2}\right) \tag{2.8}$$

Here, $\eta(x_i, t_n)$ is denoted by $\eta_i^n$, $u(x_{i+1/2}, t_n)$ is denoted by $u^n_{i+1/2}$, just as well as other variables, i.e. $H$, $u$, $\beta$, $\psi$. The length spatial grid is denoted by $\Delta x$ and the length of time discretization is denoted by $\Delta t$. In the eq. (2.8), the values of $*H$ at $\left(i+\frac{1}{2}\right)$ and $\left(i-\frac{1}{2}\right)$at are undefined, these terms are indicated by the superscript *. Following [15], the values of $*H$ is approximated using first-order upwind method as follows.

$${}^*H^n_{i+\frac{1}{2}} \begin{cases} H_i^n, & if\ u^n_{i+\frac{1}{2}} \geq 0 \\ H^n_{i+1}, & otherwise \end{cases} \tag{2.9}$$

The condition (2.9) above states that, when the flow is going to the right or $u_{i+\frac{1}{2}} \geq 0$, the information for ${}^*H_{i+\frac{1}{2}}$ values can be obtained from $H_i$. The other way around, when the flow is to the left or $u_{i+\frac{1}{2}} < 0$, the information for ${}^*H_{i+\frac{1}{2}}$ values is obtained from $H_{i+1}$. Since the coefficients $\alpha$, $\beta$, dan $\gamma$ are function of $H$, therefore the condition (2.9) are also applied for these coefficient for obtain values for $*\alpha$, $*\beta$, dan $*\gamma$.

Similar to the idea in [14], the approximation of the momentum equation of VB model (2.6) is given by the following relation

$$\frac{u^{n+1}_{i+\frac{1}{2}}-u^{n}_{i+\frac{1}{2}}}{\Delta t} = -g\left(\frac{\eta^{n+1}_{i+1}-\eta^{n+1}_{i}}{\Delta x}\right) - (u\partial_x u)^n_{i+\frac{1}{2}} \tag{2.10}$$

Rather than directly subtitute the values of $u$ in the advection term $\partial_x u$ , instead, the term is calculated by using the following relation

$$u\partial_x u = \frac{1}{h}\left(\frac{\partial(qu)}{\partial x} - u\frac{\partial q}{\partial x}\right)$$

where $q = hu$ is the horizontal momentum. A consistent approximation for the advection term as suggested in [15] is as follows

$$(u\,\partial_x u)_{i+1} = \frac{1}{\bar{H}_{i+\frac{1}{2}}}\left(\frac{\bar{q}_{i+1}.^{*}u_{i+1}-\bar{q}_i\ .^{*}u_i}{\Delta x} - u_{i+\frac{1}{2}}\frac{\bar{q}_{i+1}-\bar{q}_i}{\Delta x}\right) \tag{2.11}$$

where the bar sign denotes a simple interpolation on a point based on two points besides it, i.e.

$$\bar{H}_{i+\frac{1}{2}} = \frac{1}{2}\left(H_i \ + H_{i+1}\right), \qquad \bar{q}_i \ = \frac{1}{2}\left(q_{i+\frac{1}{2}} + q_{i-\frac{1}{2}}\right), \qquad q_{i+\frac{1}{2}} =.^{*} H_{i+\frac{1}{2}} + u_{i-\frac{1}{2}}$$

Just as $^{*}H$ in eq. (2.8), the $^{*}u$ is calculated using upwind approximation as

$$^{*}u^n_i \begin{cases} u^n_{i-\frac{1}{2}}, & if\ \bar{q}^n_i \geq 0 \\ u^n_{i+\frac{1}{2}}, & otherwise \end{cases} \tag{2.12}$$

The last equation (2.7), i.e. the elliptic equation for solving $\psi$ in every time step, is approximated by the following relation

$$\frac{^{*}\alpha_{i+\frac{1}{2}}(\psi_{i+1}-\psi_i) - \ ^{*}\alpha_{i-\frac{1}{2}}(\psi_i-\psi_{i-1})}{(\Delta x)^2} +.^{*}\gamma_i\psi_i = \frac{^{*}\beta_{i+\frac{1}{2}}u^n_{i+\frac{1}{2}} - \ ^{*}\beta_{i-\frac{1}{2}}u^n_{i-\frac{1}{2}}}{\Delta x} \tag{2.13}$$

Note that in eq. (2.13) leads into a linear matrix system that has to be solved every time step in order to obtain value of $\psi$ for a given value of $u$. The resulting matrix system is a tridiagonal matrix system that can be solved efficiently using Thomas' algorithm.

## 4. Test Cases

To test the accuracy and consistency of numerical implementation of VB model as proposed in the previous section, we use two test cases, i.e. a dispersive standing wave in a close basin and a propagation of solitary wave. Results of simulation of both cases will be compared with analytical solutions. To show the importance of dispersive effects, we compare the simulations performed with VB model with non-dispersive Shallow Water Equations (hydrostatic model) in Stelling & Duinmeijer (2003) [15].

### *4.1. Standing wave in a close basin*

The first case to test consistency of dispersive effect of VB model, we simulate a standing wave in a close basin. Computational domain that is used is $x \in [0,20]$ $m$ with a hardwall boundary condition on both sides. In the domain, we use an initial wave condition

$$\eta(x, t = 0) = 0.1\cos(k_0 x) \tag{2.14}$$

With wave number $k_0 = \pi/20$ above depth of $h_0 = 10m$. Notice that the amplitude is small compared to the depth, therefore this case is actually a weakly nonlinear problem. On the other hand, this is a dispersive case, since $k_0 h_0 = \pi/2 = 1.57$. The exact solution of the standing wave is given in the following expression

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$$\eta(x, t = 0) = 0.05[\cos(k_0(x - ct)) + \cos(k_0(x + ct))] \quad (2.15)$$

where $c$ is the exact phase velocity, defined as

$$c = \frac{\omega}{k_0} = \sqrt{\frac{g}{k_0}\tanh(k_0 h_0)}$$

For computation of the standing wave, we use $\Delta x = 0.05m$ and $\Delta t = 0.01s$. The grid size is chosen to represent the wave to be simulated. Snapshot of the simulation at t=*0s, 1s, 1.5s* and *2.5s* are shown in Figure 2.

To compare results of simulation with analytical solution as well as non-dispersive SWE, we extract a signal at $x$=1$m$. The comparison is shown in Figure 3. It can be seen that the simulation of VB model give a good agreement with the analytical solution in (2.15), whereas the simulation with non-dispersive SWE gives wrong result. The SWE has faster phase velocity, i.e. $c_0 = \sqrt{gh_0}$ , which causing the propagation of standing wave of SWE is too fast.

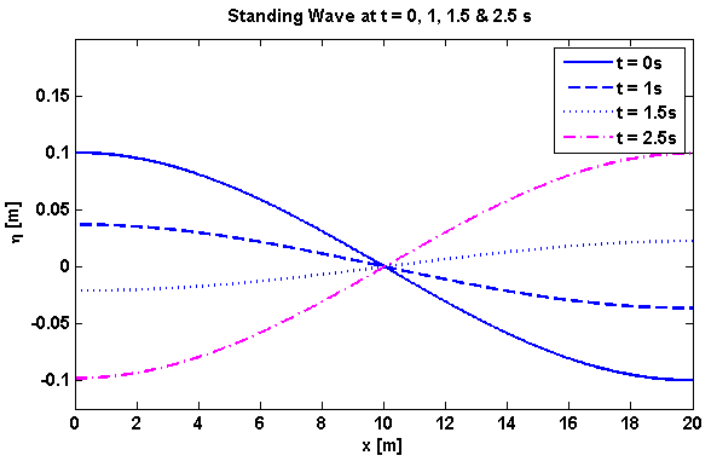


**Figure 2.** Snapshot of surface elevation at various times for standing wave simulation.

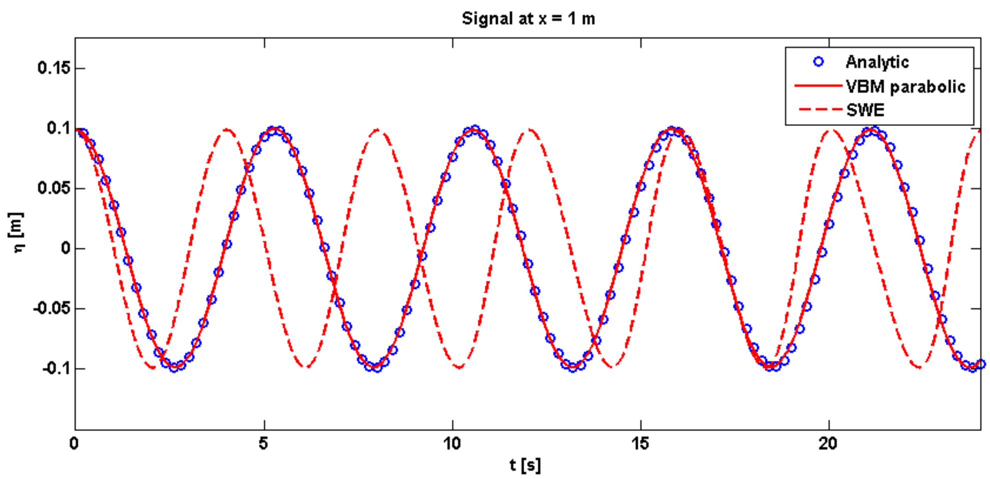


**Figure 3.** Comparison of signal at $x$=1$m$ of analytical solution (dotted blue circles), VB model (solid red line) and SWE (dashed red line) for standing wave simulation.

### *4.2. Solitary wave propagation*

The second test case is a propagation of a solitary wave. Solitary wave is a wave that propagates with a non-disturbed wave form, which occurs when the effect of dispersion is balanced with the nonlinear effect. Therefore, to be able to simulate solitary wave accurately, the wave model (as well as its numerical implementation) should has a correct dispersion as well as nonlinearity. The exact solitary wave is given by the following expression

$$\eta(x,t) = A_0\,\mathrm{sech}^2\big(\zeta(x-\lambda t)\big) \quad , \quad \lambda = \sqrt{g(h_0 + A\,)}, \quad \zeta = \sqrt{\frac{3A_0}{4h_0(h_0+A_0)}}$$

Here, $A_0$ is the amplitude of the solitary wave, $\lambda$ is the speed of the solitary wave. The solitary wave above is obtained from analytical solution of KdV equation.

We consider a domain of $x \in [0, 450]\ m$ that is discretized with $\Delta x = 1m$. The length of time discretization is $\Delta t = 0.01s$. In Figure 4, snapshot of solitary wave propagation is at $t$=14$s$, 21$s$ and 28$s$, for SWE model (left plot) and for VB model (right plot). Both numerical simulations are compared with the analytical solution. As the SWE has no dispersion effect, the nonlinearity of SWE become too dominant, as a result the solitary wave becomes steeper, nearly breaking. On the other hand, results of simulation with VB model show a good agreement with analytical solution. We also compare signals that are extracted at $x$=250$m$ as shown in Figure 5. Just as in the snapshot plot, similar patterns are also seen in these signals, i.e. the SWE result shows steeper wave as the dispersion of the model is absent.

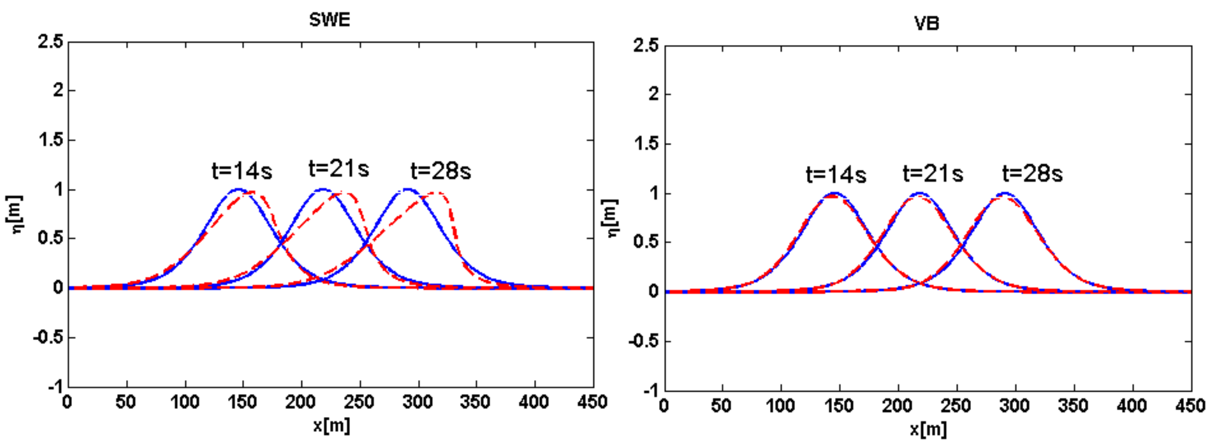


**Figure. 4.** Snapshot of simulation with SWE model (left plot, denoted by red dashed line) and with VB model (right plot, denoted by red dashed line) with analytical solution (solid blue line) at $t$=14s, 21s, and 28s.

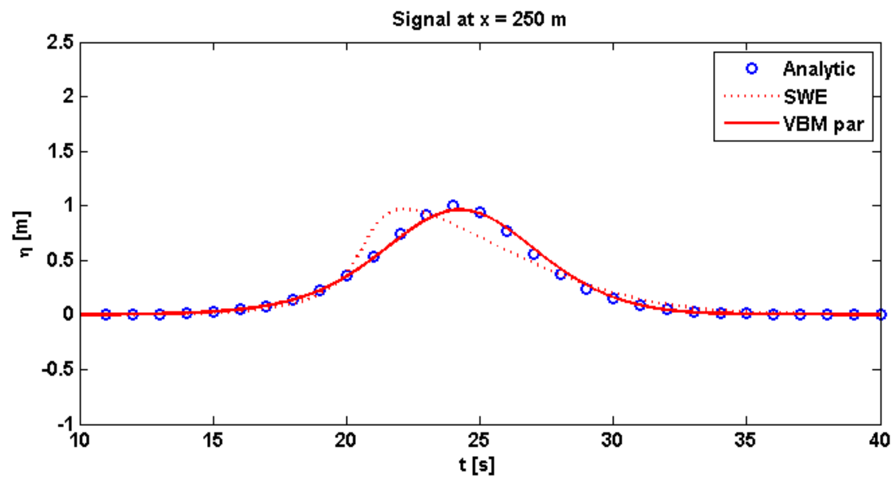


**Figure. 5.** Comparison of signal at $x$=250$m$ of analytical solution of solitary wave (blue circles), simulation with SWE (dotted red line) and simulation with VB model (solid red line).

## 5. Conclusion and discussion

The original idea of the staggered grid introduced by Stelling & Duinmeijer (2003) [15] is extended for solving numerically the dispersive wave model, i.e. the Variational Boussinesq (VB) model. An additional elliptic equation in the VB model leads to a tridiagonal matrix system that is solved efficiently by using Thomas' algorithm. Two test cases are presented to test consistency of numerical implementation of the VB model, i.e. standing wave simulation and solitary wave propagation. Both cases show the importance of dispersion property of wave model. The comparison of the results of simulation with staggered grid VB model show a good agreement with analytical solution, whereas the non-dispersive staggered grid Shallow Water Equations (SWE) cannot simulate the proposed phenomena accurately. The staggered grid implementation for the 1D VB model here can be extended into 2D problem.

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