Fakultas Informatika
School of Computing
Telkom University

CNH2B4 / KOMPUTASI NUMERIK

TIM DOSEN

KK MODELING AND COMPUTATIONAL EXPERIMENT

4 SOLUSI SISTEM PERSAMAAN LINEAR: METODE ELIMINASI GAUSS & GAUSS-JORDAN

SISTEM PERSAMAAN LINEAR

- Sistem Linier (sistem banyak variabel) memiliki persamaan umum:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\
&\ldots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

- Dalam bentuk perkalian matrik menjadi:

$$
\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_n \end{bmatrix}}_{x}
=
\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ b_n \end{bmatrix}}_{b}
$$

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Eliminasi Gauss

- Sistem persamaan linear:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ &\ldots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

- Matriks yang diperluas:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & \vdots & a_{1,n+1} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & \vdots & a_{2,n+1} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} & \vdots & a_{3,n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} & \vdots & a_{n,n+1} \end{bmatrix}$$

- Operasi baris elementer:

$$(E_j - (a_{ji}/a_{ii})E_i) \rightarrow (E_j) \qquad j = i+1, i+2, \ldots, n$$

sampai terbentuk matriks segitiga atas.

Eliminasi Gauss

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,n+1} \\ a_{2,n+1} \\ a_{3,n+1} \\ \dots \\ a_{n,n+1} \end{bmatrix} \xrightarrow{\textbf{OBE}} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \dots & & & & \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,n+1} \\ a_{2,n+1} \\ a_{3,n+1} \\ \dots \\ a_{n,n+1} \end{bmatrix}$$

Solusinya menggunakan substitusi mundur:

$$a_{nn}x_n = a_{n,n+1} \rightarrow x_n = a_{n,n+1} / a_{nn}$$

$$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = a_{n-1,n+1} \rightarrow x_{n-1} = \frac{a_{n-1,n+1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$a_{n-2,n-2}x_{n-2} + a_{n-2,n-1}x_{n-1} + a_{n-2,n}x_n = a_{n-2,n+1} \rightarrow x_{n-2} = \frac{a_{n-2,n+1} - a_{n-2,n-1}x_{n-1} - a_{n-2,n}x_n}{a_{n-2,n-2}}$$

.....dst

Sekali x_n, x_{n-1}, x_{n-2}, …, x_{i+1} diketahui , maka nilai x_i dapat dihitung dengan:

$$x_i = \frac{(a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}\, x_j)}{a_{ii}}$$

dengan i = n-1, n-2, …, 1
dan $a_{ii} \neq 0$

Gaussian Elimination with Backward Substitution

To solve the $n \times n$ linear system

$$\begin{aligned}
E_1 &: \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1,n+1} \\
E_2 &: \quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = a_{2,n+1} \\
&\ \ \vdots \\
E_n &: \quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = a_{n,n+1}
\end{aligned}$$

INPUT number of unknowns and equations n; augmented matrix $A = [a_{ij}]$, where $1 \le i \le n$ and $1 \le j \le n+1$.

OUTPUT solution $x_1, x_2, \ldots, x_n$ or message that the linear system has no unique solution.

Step 1 For $i = 1, \ldots, n-1$ do Steps 2–4. (*Elimination process.*)

Step 2 Let p be the smallest integer with $i \le p \le n$ and $a_{pi} \ne 0$.
If no integer p can be found
then OUTPUT ('no unique solution exists');
STOP.

Step 3 If $p \ne i$ then perform $(E_p) \leftrightarrow (E_i)$.

Step 4 For $j = i+1, \ldots, n$ do Steps 5 and 6.

Step 5 Set $m_{ji} = a_{ji}/a_{ii}$.

Step 6 Perform $(E_j - m_{ji}E_i) \to (E_j)$;

Step 7 If $a_{nn} = 0$ then OUTPUT ('no unique solution exists');
STOP.

Step 8 Set $x_n = a_{n,n+1}/a_{nn}$. (*Start backward substitution.*)

Step 9 For $i = n-1, \ldots, 1$ set $x_i = \left[a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j\right] \Big/ a_{ii}$.

Step 10 OUTPUT $(x_1, \ldots, x_n)$; (*Procedure completed successfully.*)
STOP.

Burden, Richard L., and J. Douglas Faires. *Numerical Analysis*. Brooks/Cole, *USA*, 2001.

Eliminasi Gauss (Cont.)

- Contoh :

$$2x_1 + 3x_2 - x_3 = 5$$
$$4x_1 + 4x_2 - 3x_3 = 3$$
$$-2x_1 + 3x_2 - x_3 = 1$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right] \begin{array}{c} E_2 - \frac{4}{2}E_1 \\ E_3 - \frac{-2}{2}E_1 \end{array} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 6 & -2 & 6 \end{array}\right] E_3 - \frac{6}{-2}E_2 \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

- Subsitusi Mundur :

$$x_3 = \frac{-15}{-5} = 3$$

$$x_2 = \frac{-7-(-1)x_3}{-2} = \frac{-7-(-1)(3)}{-2} = 2$$

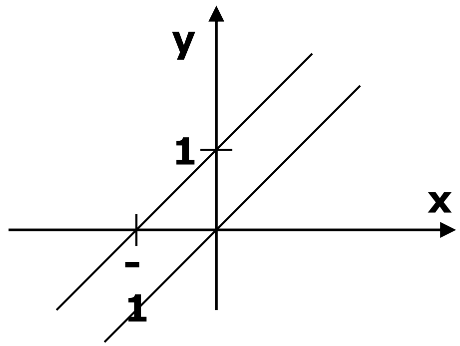
$$x_1 = \frac{5-[3x_2+(-1)x_3]}{2} = \frac{5-[3(2)+(-1)(3)]}{2} = 1$$

Eliminasi Gauss (Cont.)

- pivot bernilai nol diatasi dengan Strategi Pivoting:

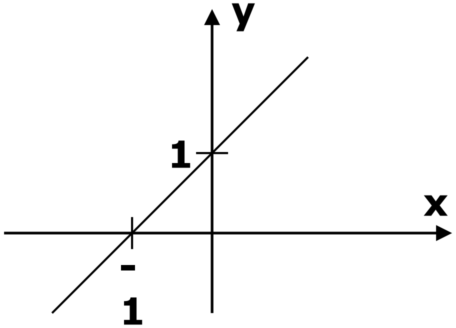
→ jika $a_{pp} = 0$, cari baris k yang $a_{k,p} \neq 0$ dan $k>p$, kemudian pertukarkan baris p dengan baris k.

Kemungkinan Solusi SPL



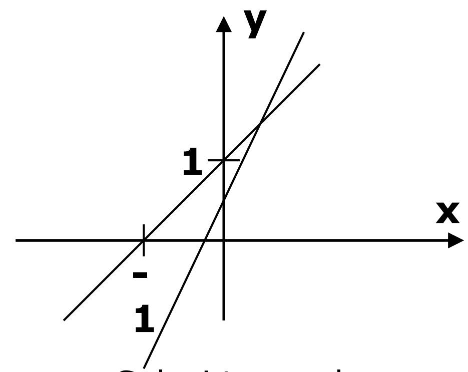
Tidak ada solusi

$$\begin{bmatrix} 1 & 1 & 2 & | & 4 \\ 2 & -1 & 1 & | & 2 \\ 1 & 2 & 3 & | & 7 \end{bmatrix} \xrightarrow{\text{Eliminasi Gauss}} \begin{bmatrix} 1 & 1 & 2 & | & 4 \\ 0 & -3 & -3 & | & -6 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$



Solusi banyak

$$\begin{bmatrix} 1 & 1 & 2 & | & 4 \\ 2 & -1 & 1 & | & 2 \\ 1 & 2 & 3 & | & 6 \end{bmatrix} \xrightarrow{\text{Eliminasi Gauss}} \begin{bmatrix} 1 & 1 & 2 & | & 4 \\ 0 & -3 & -3 & | & -6 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$



Solusi tunggal

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 2 & 3 & 1 & | & 1 \\ 3 & 1 & 2 & | & 1 \end{bmatrix} \xrightarrow{\text{Eliminasi Gauss}} \begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 1 & -1 & | & 1 \\ 0 & 0 & -3 & | & 3 \end{bmatrix}$$

Eliminasi Gauss-Jordan

- Format matrik mengalami perubahan :

  $Ax = b \rightarrow I\,x = b'$

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_3 \\ \dots & & & & & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & b_n \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & \dots & 0 & b_1' \\ 0 & 1 & 0 & \dots & 0 & b_2' \\ 0 & 0 & 1 & \dots & 0 & b_3' \\ \dots & & & & & \dots \\ 0 & 0 & 0 & \dots & 1 & b_n' \end{array}\right]$$

- Matrik A bersamaan dengan vektor b dieliminasi sampai matrik A menjadi matrik Identitas
- solusinya :

  $x_1 = b_1', x_2 = b_2', \dots\dots x_n = b_n'$

- Contoh

$$2x_1 + 3x_2 - x_3 = 5$$
$$4x_1 + 4x_2 - 3x_3 = 3$$
$$-2x_1 + 3x_2 - x_3 = 1$$

- Jawab menggunakan eliminasi Gauss-Jordan

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right] \begin{array}{c} E_2 - \frac{4}{2}E_1 \\ E_3 - \frac{-2}{2}E_1 \end{array} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 6 & -2 & 6 \end{array}\right] E_3 - \frac{6}{-2}E_2 \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

$$\begin{array}{c} E_2 - \frac{-1}{-5}E_3 \\ E_1 - \frac{-1}{-5}E_3 \end{array} \left[\begin{array}{ccc|c} 2 & 3 & 0 & 8 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & -5 & -15 \end{array}\right] E_1 - \frac{3}{-2}E_2 \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & -5 & -15 \end{array}\right] \begin{array}{c} \frac{1}{2}E_1 \\ \frac{1}{-2}E_2 \\ \frac{1}{-5}E_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$x_1 = 1, x_2 = 2, x_3 = 3$$

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THANK YOU